21/9/23

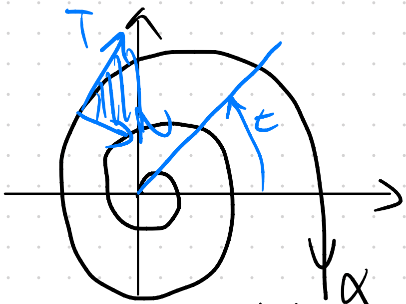
MATH 4030 Tutorial

Announcements:

HW1 posted on course website. Submit on Gradescope. Due 2/10 11:59pm.

Q1: Compute the arc-length, curvature, torsion of the Logarithmic spiral given by $\alpha: I \to \mathbb{R}^3$,

$$\alpha(t) = (ae^{bt}\cos t,\ ae^{bt}\sin t,\ 0), \quad a>0,\ b<0 \text{ constants.}$$



Soln: α is a plane curve, so $\tau = 0$.

$$s(t) = \int_0^t |\alpha'(u)|\,du.$$

$$\alpha'(u) = (abe^{bu}\cos u - ae^{bu}\sin u,\ abe^{bu}\sin u + ae^{bu}\cos u,\ 0)$$

some algebra

$$|\alpha'(u)|^2 \overset{\downarrow}{=} a^2b^2e^{2bu}\underbrace{(\cos^2 u + \sin^2 u)}_{1} + a^2e^{2bu}\underbrace{(\cos^2 u + \sin^2 u)}_{1}$$

$$= a^2e^{2bu}(1+b^2).$$

$$|\alpha'(u)| = ae^{bu}\sqrt{1+b^2}. \qquad s(t) = \int_0^t ae^{bu}\sqrt{1+b^2}\,du = a\sqrt{1+b^2}\int_0^t e^{bu}\,du$$

$$= \frac{a}{b}\sqrt{1+b^2}\,e^{bu}\Big|_0^t = \frac{a}{b}\sqrt{1+b^2}\,(e^{bt}-1).$$

$$k(t) = \frac{|\alpha'(t) \times \alpha''(t)|}{|\alpha'(t)|^3}.$$

$$\alpha''(t) = (ab^2e^{bt}\cos t - 2abe^{bt}\sin t - abe^{bt}\cos t,$$
$$ab^2e^{bt}\sin t + 2abe^{bt}\cos t - ae^{bt}\sin t,\ 0).$$

$$\alpha'(t)\times\alpha''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ abe^{bt}\cos t - ae^{bt}\sin t & abe^{bt}\sin t + ae^{bt}\cos t & 0 \\ ab^2e^{bt}\cos t - 2abe^{bt}\sin t - abe^{bt}\cos t & ab^2e^{bt}\sin t + 2abe^{bt}\cos t - ae^{bt}\sin t. & 0 \end{vmatrix}$$

$$= a^2(1+b^2)e^{2bt}\hat{k}$$

$|\alpha'(t)\times\alpha''(t)| = a^2(1+b^2)e^{2bt}$. $\quad |\alpha'(t)|^3 = a^3e^{3bt}(1+b^2)^{3/2}$

$$k(t) = \frac{1}{ae^{bt}\sqrt{1+b^2}}.$$

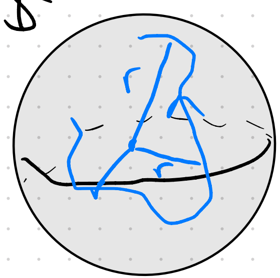
Q2: Suppose $\alpha : I \to \mathbb{R}^3$ w/ $\tau \neq 0$, $k' \neq 0$ for all $s \in I$ and α lies on a sphere. Show that

$$\rho^2 + (\rho')^2\lambda^2 = \text{const.}$$

where $\rho = \frac{1}{k}$, $\lambda = \frac{1}{\tau}$ Hint: differentiate $|\alpha|$ three times to obtain

$$\alpha = -\rho N - \rho'\lambda B.$$

S^2



Pf: α lies on a sphere, so $\exists\ r > 0$ s.t.

$$|\alpha(s)|^2 = r^2 \text{ for all } s \in I.$$

Differentiating, we have

$$0 = \frac{d}{ds}|\alpha(s)|^2 = 2\overset{\langle \alpha', \alpha \rangle}{\overset{\shortparallel}{\alpha' \cdot \alpha}} \Rightarrow \alpha' \cdot \alpha = 0 \quad (\alpha \cdot T = 0).$$

Differentiating again, we have

$$0 = \alpha' \cdot \alpha' + \alpha'' \cdot \alpha = \underbrace{|\alpha'(s)|^2}_{\substack{\shortparallel \\ 1}} + \alpha'' \cdot \alpha \Rightarrow \alpha'' \cdot \alpha = -1$$

arc-length param.

$$\underset{kN \cdot \alpha = -1.}{\shortparallel}$$

$$\Rightarrow N \cdot \alpha = \frac{-1}{k} = -\rho.$$

Differentiating again, we have

$$0 = 2\alpha'' \cdot \alpha' + \alpha''' \cdot \alpha + \alpha'' \cdot \alpha' = \overset{0}{\cancel{3\alpha'' \cdot \alpha'}} + \alpha''' \cdot \alpha$$

$$\Rightarrow \alpha''' \cdot \alpha = 0.$$

$$\Rightarrow 0 = (kN)' \cdot \alpha = k' N \cdot \alpha + k(-kT + \tau B) \cdot \alpha$$

$$= -k^2 \overset{0}{\cancel{T \cdot \alpha}} + k' \underbrace{N \cdot \alpha}_{-\frac{1}{k}} + k\tau B \cdot \alpha$$

$$= -\frac{k'}{k} + k\tau B\cdot\alpha$$

$$\Rightarrow B\cdot\alpha = \frac{k'}{k^2\tau} = -\rho'\lambda$$

So $\alpha = -\rho N - \rho'\lambda B$.

Again, since α lies on a sphere,

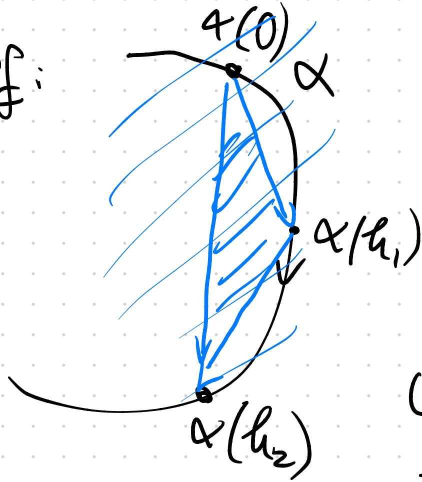
$$\text{const} = |\alpha(s)|^2 = (-\rho)^2 + (-\rho'\lambda)^2 = \rho^2 + (\rho')^2\lambda^2.$$

Note: $\rho^2 + (\rho')^2\lambda^2 = \text{const} \Rightarrow \alpha$ lies on a sphere.

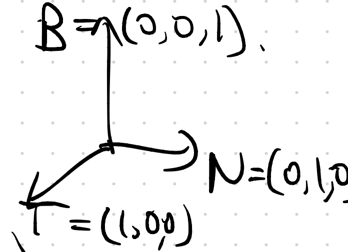
$$\alpha - (-\rho N - \rho'\lambda B) = \text{const}.$$

Q3: $\alpha : I \to \mathbb{R}^3$ param. by arc-length, $k \neq 0$. Using the local canonical form, show that the osculating plane is the limit position of planes passing through $\alpha(s), \alpha(s+h_1), \alpha(s+h_2)$ as $h_1, h_2 \to 0$.

Pf:



By the local canonical form



$$\alpha(s) = (x(s), y(s), z(s)) + o(s^3)$$

$$= \left(s - \frac{k^2 s^3}{6}, \frac{k}{2}s^2 + \frac{k' s^3}{6}, \frac{-k\tau}{6} s^3\right) + o(s^3).$$

Let $ax + by + cz = 0$ be the plane passing through $\alpha(0)$, $\alpha(h_1)$, $\alpha(h_2)$.

$F(s) = ax(s) + by(s) + cz(s)$, note $F(0) = F(h_1) = F(h_2) = 0$.

$F'(0) = a$

$F''(0) = bk.$

By MVT x2, $\exists\, c_1 \in (0, h_1)$, $c_2 \in (h_1, h_2)$ s.t.

s.t. $F'(c_1) = \dfrac{F(h_1) - F(0)}{h_1} = 0$

$F'(c_2) = \dfrac{F(h_2) - F(h_1)}{h_2 - h_1} = 0$

This shows $a \to 0$ as $h_1, h_2 \to 0$.

Using MVT again will show $bh \to 0$ as $h_1, h_2 \to 0$.

$\Rightarrow b \to 0$.

Then taking $h_1, h_2 \to 0$, $F(s) \to Cz = 0$. //.